away) is extremely complex. Conversely, when the mode of fluid induction is established, then there exists the possibility of an appreciable change of the electric characteristics of the jet with the aid of guiding surfaces inserted into the flow. Examples of such control by means of the electrical jet were investigated in [6].

The data obtained are applicable to a quantitative prediction of the effects of control by means of the electric jet in each specific case. Let us also point out that instead of the parameter β sometimes it appears more convenient to introduce the value of the nondimensional current imposed in the entrance section.

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ON THE GENERAL THEORY OF ALMOST SELF-SIMILAR NONSTATIONARY FLOWS

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A system of variational equations is considered, to which nonstationary perfect gas flows differing slightly from the self-similar ones are subject. General expressions are written for the mass, energy and momentum of the material within the perturbed domain and the time-independent summands are extracted therefrom. The first integrals of the variational equations which are extremely simple in form correspond to these summands. The arbitrary constants are selected in such a way that the boundary conditions on the front of a strong shock wave are satisfied automatically.

1. Let us assume that the nonstationary motion of a perfect gas is caused by an explosion or the expansion of a piston. Let the equation giving the position $r_2\ (t,\ \varphi,\ \vartheta)$ of the shock wave propagated over the initially cold gas at rest be represented for large values of the time t as

$$r_2 = (bt)^n \left(1 + \varepsilon t^{-2m/(\nu+2)} R_2 + \ldots\right) \tag{1.1}$$

Here b is a dimensional constant, ε, n and m are some positive numbers, where $\varepsilon \ll 1$, the parameter ν takes on the values $1-3$ depending on the dimensionality of the problem, the quantity R_2 can be either constant or a function of the angular variables φ, ϑ. Let v_n and v_τ denote the components of the velocity vector normal and tangential to the surface of strong discontinuity, ρ is the density, p is the pressure, $\varkappa$ is the ratio between the specific heats, and N is the rate of shock wave displacement. If the state of the gas directly ahead of the front is noted by the subscript 1 and directly behind it by the subscript 2, then the Rankine-Hugoniot conditions become

$$v_{n2} = \frac{2}{\varkappa+1} N, \qquad v_{\tau 2} = 0, \qquad \rho_2 = \frac{\varkappa+1}{\varkappa-1}\rho_1, \qquad p_2 = \frac{2}{\varkappa+1}\rho_1 N^2 \tag{1.2}$$

2. Let us first consider $R_2 = \text{const} = 1$. Then the solution of the problem of gas motion behind the shock wave (1.1) can be sought as series in decreasing powers of t with coefficients which are functions of the variable $\lambda = r/(bt)^n$. The velocity vector will have only a nonzero radial component v_r, hence

$$\begin{aligned}
v_r &= \frac{2n}{\varkappa+1} b^n t^{n-1} \left[f(\lambda) + \varepsilon t^{-2m/(\nu+2)} f_m(\lambda) + \ldots\right] \\
\rho &= \frac{\varkappa+1}{\varkappa-1}\rho_1 \left[g(\lambda) + \varepsilon t^{-2m/(\nu+2)} g_m(\lambda) + \ldots\right] \\
p &= \frac{2n^2}{\varkappa+1}\rho_1 b^{2n} t^{2(n-1)} \left[h(\lambda) + \varepsilon t^{-2m/(\nu+2)} h_m(\lambda) + \ldots\right]
\end{aligned} \tag{2.1}$$

The first approximation functions f, g and h give the characteristics of self-similar flows for the study of which Sedov [1] indicated the general approach. The system of ordinary differential equations governing them is contained in [2]. Substituting the expansion (2.1) into the Euler equations

$$\begin{gathered}
\frac{\partial\rho}{\partial t} + \frac{\partial \rho v_r}{\partial r} + (\nu-1)\frac{\rho v_r}{r} = 0 \\
\rho\frac{\partial v_r}{\partial t} + \rho v_r \frac{\partial v_r}{\partial r} + \frac{\partial p}{\partial r} = 0 \\
\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \varkappa p\left[\frac{\partial v_r}{\partial r} + (\nu-1)\frac{v_r}{r}\right] = 0
\end{gathered} \tag{2.2}$$

we derive a system of three ordinary differential equations for the second approximation functions f_m, g_m and h_m. Namely

$$\begin{gathered}
g\frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dg_m}{d\lambda} + \left(\frac{dg}{d\lambda} + \frac{\nu-1}{\lambda}g\right)f_m + \\
\left[\frac{df}{d\lambda} + \frac{\nu-1}{\lambda}f - \frac{m(\varkappa+1)}{n(\nu+2)}\right]g_m = 0 \\
g\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{df_m}{d\lambda} + \frac{\varkappa-1}{2}\frac{dh_m}{d\lambda} + \left[\frac{df}{d\lambda} + \frac{\varkappa+1}{2n}\left(n - 1 - \frac{2m}{\nu+2}\right)\right]gf_m + \\
\left[\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{df}{d\lambda} + \frac{(n-1)(\varkappa+1)}{2n}f\right]g_m = 0
\end{gathered}$$

$$\varkappa h \frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dh_m}{d\lambda} + \left[\frac{dh}{d\lambda} + \frac{(\nu-1)\varkappa}{\lambda} h\right] f_m + \left[\varkappa \frac{df}{d\lambda} + \frac{(\nu-1)\varkappa}{\lambda} f + \frac{\varkappa+1}{n}\left(n - 1 - \frac{m}{\nu+2}\right)\right] h_m = 0 \tag{2.3}$$

The initial values of these functions are established by using the conditions (1.2) for $\lambda = 1$

$$f_m = \frac{1}{n}\left(4n - 3 - \frac{2m}{\nu+2}\right) + \frac{2(\nu-1)\varkappa}{\varkappa+1}$$

$$g_m = \frac{6(n-1)}{n(\varkappa-1)} + \frac{2(\nu-1)}{\varkappa+1} \tag{2.4}$$

$$h_m = \frac{2}{n}\left[\frac{(3n-2)\varkappa - 2n + 1}{\varkappa - 1} - \frac{2m}{\nu+2}\right] + \frac{2(\nu-1)\varkappa}{\varkappa+1}$$

Using the fact that the entropy at each particle does not change after the shock wave has passed, Lidov [3] obtained the final integral of (2.3). Its natural extension is due to Korobeinikov [4] who studied higher approximations in the small perturbations method. Let us indicate the mode which for some values of m permits writing down other integrals of the equations under consideration which correspond to the laws of gas mass, energy, and momentum conservation in the perturbed motion domain. Upon going over to the variable λ the relationship (1.1) yielding the shock front position becomes

$$\lambda_2 = 1 + \varepsilon t^{-2m/(\nu+2)} R_2 + \ldots \tag{2.5}$$

Let us compute the mass $M(\lambda_2, \lambda)$ of gas enclosed between the moving surfaces $\lambda = \lambda_2$ and $\lambda = \text{const}$. Let Σ_2 denote the former, and Σ_λ the latter. We have

$$M(\lambda_2, \lambda) = k_\nu \int_r^{r_2} \rho r^{\nu-1} dr, \qquad k_\nu = \begin{cases} 2, & \nu = 1 \\ 2\pi, & \nu = 2 \\ 4\pi, & \nu = 3 \end{cases}$$

Using the expansion (2.1), we transform the expression written down as follows:

$$M(\lambda_2, \lambda) = \frac{\varkappa+1}{\varkappa-1} k_\nu \rho_1 b^{\nu n} t^{\nu n}\left[\int_\lambda^1 g\lambda^{\nu-1} d\lambda + \varepsilon t^{-2m/(\nu+2)}\left(1 + \int_\lambda^1 g_m \lambda^{\nu-1} d\lambda\right) + \ldots\right] \tag{2.6}$$

Let $\Sigma = \Sigma_2 \cup \Sigma_\lambda$, then by virtue of the Rankine-Hugoniot conditions the total time derivative of the mass of gas will be

$$\frac{dM(\lambda_2, \lambda)}{dt} = \oint_\Sigma \rho (N_\sigma - v_n) d\sigma = k_\nu [\rho_1 N r_2^{\nu-1} - \rho (v_\lambda - v_r) r^{\nu-1}]$$

Here N_σ is the rate of displacement of an element $d\sigma$ along its normal, v_λ is the rate of expansion of the inner surface Σ_λ. Evidently $N_\sigma = -v_\lambda$ for all its points. Furthermore, we have

$$\frac{dM(\lambda_2, \lambda)}{dt} = n k_\nu \rho_1 b^{\nu n} t^{\nu n - 1}\left\{1 - \frac{\varkappa+1}{\varkappa-1}\lambda^\nu g + \frac{2}{\varkappa-1}\lambda^{\nu-1} fg + \varepsilon t^{-2m/(\nu+2)}\left[\nu - \frac{2m}{n(\nu+2)} - \frac{\varkappa+1}{\varkappa-1}\lambda^\nu g_m + \frac{2}{\varkappa-1}\lambda^{\nu-1}(g f_m + f g_m)\right] + \ldots\right\} \tag{2.7}$$

Let us differentiate (2.6) and equate the relation obtained for $dM(\lambda_2, \lambda)/dt$ to the right side of (2.7). The main terms proportional to $t^{\nu n-1}$ yield

$$\frac{\nu(\varkappa+1)}{2}\int_{\lambda}^{1}\lambda^{\nu-1}g\,d\lambda = \frac{\varkappa-1}{2}+\lambda^{\nu-1}\left(f-\frac{\varkappa+1}{2}\lambda\right)g$$

This equality defines the mass of gas in a volume bounded by the shock wave front and any surface $\lambda = \mathrm{const}$ in the main self-similar flow. Considering terms of order ε and assuming

$$m = {}^1/_2\,\nu\,(\nu+2)\,n$$

we have analogously for the variational functions

$$\lambda g_m - \frac{2}{\varkappa+1}(gf_m + fg_m) = 0 \tag{2.8}$$

There are no integral terms here since the second summand in (2.6) for $M(\lambda_2, \lambda)$ is independent of the time for the selected value of the exponent m. It is easy to verify that the equality (2.8) automatically satisfies the boundary conditions (2.4) on the surface of strong discontinuity. An arbitrary constant c_M can be introduced into this equality and given the standard form of a first integral of the ordinary differential equations system (2.3) by replacing the zero in its right side by $c_M\,\lambda^{1-\nu}$

Let us calculate the energy $E(\lambda_2, \lambda)$ of the mass of gas under consideration. Evidently

$$E(\lambda_2, \lambda) = k_\nu\int_{r}^{r_2}\left(\frac{\rho v_r^2}{2}+\frac{p}{\varkappa-1}\right)r^{\nu-1}\,dr$$

Substituting the expansion (2.1) here, we find

$$E(\lambda_2, \lambda) = \frac{2n^2}{\varkappa^2-1}k_\nu\rho_1 b^{(\nu+2)n}t^{(\nu+2)n-2}\left\{\int_{\lambda}^{1}(f^2g+h)\,\lambda^{\nu-1}\,d\lambda + \right.$$
$$\left.\varepsilon t^{-2m/(\nu+2)}\left[2+\int_{\lambda}^{1}(2fgf_m+f^2g_m+h_m)\,\lambda^{\nu-1}\,d\lambda\right]+\ldots\right\} \tag{2.9}$$

On the other hand, the total derivative of the energy $E(\lambda_2, \lambda)$ with respect to the time is determined as

$$\frac{dE(\lambda_2, \lambda)}{dt} = \oint_{\Sigma}\left[(N_\sigma - v_n)\left(\frac{\rho v_r^2}{2}+\frac{\varkappa p}{\varkappa-1}\right)-pN_\sigma\right]d\sigma =$$
$$-k_\nu\left[(v_\lambda - v_r)\left(\frac{\rho v_r^2}{2}+\frac{\varkappa p}{\varkappa-1}\right)-pv_\lambda\right]r^{\nu-1}$$

from which there readily follows

$$\frac{dE(\lambda_2, \lambda)}{dt} = -\frac{2n^3}{\varkappa^2-1}k_\nu\rho_1 b^{(\nu+2)n}t^{(\nu+2)n-3}\left\{\lambda^\nu(f^2g+h) - \right.$$
$$\frac{2}{\varkappa+1}\lambda^{\nu-1}f(f^2g+\varkappa h)+\varepsilon t^{-2m/(\nu+2)}\left[\lambda^\nu(2fgf_m+f^2g_m+h_m) - \right.$$
$$\left.\left.\frac{2}{\varkappa+1}\lambda^{\nu-1}((3f^2g+\varkappa h)f_m+f^3g_m+\varkappa fh_m)\right]+\ldots\right\}$$

A comparison between the expression obtained and the equation which results from differentiating (2.9) yields for the first approximation functions

$$\frac{1}{2n}(\varkappa+1)\,[(\nu+2)n-2]\int_{\lambda}^{1}(f^2g+h)\,\lambda^{\nu-1}\,d\lambda =$$

$$= \lambda^{\nu-1}\left[(f^2g + h)\left(f - \frac{\varkappa+1}{2}\lambda\right) + (\varkappa - 1) fh\right]$$

This relationship permits finding the gas energy in any domain of the initial self-similar flow between the shock wave and the surface λ = const. The exception is just the case with $n = 2 / (\nu + 2)$, when it is reduced to the final integral in the strong explosion problem, established in [5, 6]. Now, let the exponent

$$m = {}^1/_2(\nu + 2)\,[(\nu + 2)\,n - 2]$$

when the term proportional to ε in (2.9) becomes a constant. It yields no contribution to the derivative $dE\ (\lambda_2,\ \lambda)\ /\ dt$, hence, we have the following integral for the variational functions

$$\lambda\,(2fgf_m + f^2g_m + h_m) - \frac{2}{\varkappa+1}\,[(3f^2g + \varkappa h)\,f_m + f^3g_m + \varkappa f h_m] = 0 \quad (2.10)$$

which automatically satisfies the conditions (2.4) on the strong shock wave front. It is easy to verify that (2.10) continues to be a result of the system (2.3) if the zero in its right side is replaced by $c_E\,\lambda^{1-\nu}$, where the constant c_E can be selected arbitrarily.

3. Let us turn to seeking the integral for the functions f_m, g_m and h_m which is related to the momentum conservation law in the perturbed flow domain. Since the momentum is a vector, we can not consider the quantity R_2 from the expansions (1.1) and (2.5) for the shock front as independent of the angles φ and ϑ; otherwise, the total momentum of the gas would vanish. Only plane waves whose asymmetry can occur even for R_2 = const, are an exception.

Let us direct the z-axis of a Cartesian coordinate system along the momentum vector. Let V denote the gas volume bounded by the expanding surfaces Σ_2 and Σ_λ introduced above. The integral containing one v_z component of the particle velocity yields a contribution to the total momentum $I\ (\lambda_2,\ \lambda)$, hence

$$I\,(\lambda_2,\,\lambda) = \int\limits_V \rho v_z\,dV \quad (3.1)$$

and by virtue of the Rankine-Hugoniot conditions its time-derivative is

$$\frac{dI\,(\lambda_2,\,\lambda)}{dt} = \oint\limits_{\Sigma_\lambda} [\rho v_z\,(N_\sigma - v_n) - pn_z]\,d\sigma \quad (3.2)$$

Here n_z is the projection of the unit normal on the z-axis. Formulas (3.1) and (3.2) can be used to solve problems with any number of space dimensions, however, it is more convenient to investigate one-, two- and three-dimensional flows separately.

For plane flows the parameter is $\nu = 1$, $v_z = v_r$ and R_2 = const = 1. In this simplest case

$$I\,(\lambda_2,\,\lambda) = \frac{2n}{\varkappa-1}\,\rho_1 b^{2n}\,t^{2n-1}\left\{\int\limits_\lambda^1 fg\,d\lambda + \varepsilon t^{-{}^3/_2 m}\left[1 + \int\limits_\lambda^1 (gf_m + fg_m)\,d\lambda\right] + \ldots\right\} \quad (3.3)$$

Substituting the expansion (2.1) in the equality (3.2), we find

$$\frac{dI\,(\lambda_2,\,\lambda)}{dt} = -\frac{2n^2}{\varkappa-1}\,\rho_1 b^{2n}\,t^{2(n-1)}\left\{\lambda fg - \frac{2}{\varkappa+1}\left(f^2g + \frac{\varkappa-1}{2}\,h\right) + \right.$$
$$\left.\varepsilon t^{-{}^3/_2 m}\left[\lambda\,(gf_m + fg_m) - \frac{2}{\varkappa+1}\left(2fgf_m + f^2g_m + \frac{\varkappa-1}{2}\,h_m\right)\right] + \ldots\right\}$$

Let us compare the last relationship with the expression for the derivative $dI\ (\lambda_2,\ \lambda)\ /dt$

which can be determined by using differentiation of (3.3). We have the following equation for the principal terms:

$$\frac{(\varkappa+1)(2n-1)}{2n}\int_{\lambda}^{1} fg\,d\lambda = fg\left(f-\frac{\varkappa+1}{2}\lambda\right)+\frac{\varkappa-1}{2}h$$

which permits calculation of the momentum of a plane self-similar wave. In the exceptional case, $n={}^1/_2$, from which the final integral presented in [1] follows; it interrelates the first approximation functions. In order for the term of order ε in (3.3) to take on a constant value, it is necessary to assume

$$m={}^3/_2\,(2n-1)$$

Under this condition the variational functions form an integral which agrees with the initial data (2.4) for a strong shock front

$$\lambda(gf_m+fg_m)-\frac{1}{\varkappa+1}\left[4fgf_m+2f^2g_m+(\varkappa-1)h_m\right]=0 \qquad (3.4)$$

Let us turn to a study of waves which have cylindrical symmetry in a first approximation; they are realized when the parameter $\nu=2$. The total momentum of the gas differs from zero if the quantity R_2 is a function of the polar angle φ measured from the direction of the z-axis. Expanding $R_2(\varphi)$ in a Fourier series, we consider the term with the kth harmonic. To do this, it is sufficient to assume that $R_2=\cos(k\varphi+\alpha_k)$, where α_k is an arbitrary constant. Instead of (2.1), we write respectively

$$v_r=\frac{2n}{\varkappa+1}b^n t^{n-1}\left[f(\lambda)+\varepsilon t^{-m/2}f_m(\lambda)\cos(k\varphi+\alpha_k)+\ldots\right]$$

$$v_\varphi=\frac{2n}{\varkappa+1}\varepsilon b^n t^{n-1-m/2}u_m(\lambda)\sin(k\varphi+\alpha_k)+\ldots \qquad (3.5)$$

$$\rho=\frac{\varkappa+1}{\varkappa-1}\rho_1\left[g(\lambda)+\varepsilon t^{-m/2}g_m(\lambda)\cos(k\varphi+\alpha_k)+\ldots\right]$$

$$p=\frac{2n^2}{\varkappa+1}\rho_1 b^{2n}t^{2(n-1)}\left[h(\lambda)+\varepsilon t^{-m/2}h_m(\lambda)\cos(k\varphi+\alpha_k)+\ldots\right]$$

It is understood that the Euler equations (2.2) must be supplemented by terms dependent on φ and containing a v_φ velocity vector component. The functions f, g and h satisfy the previous system of ordinary differential equations. As regards the second-approximation functions, we then derive the following system to determine them as a result of substituting the expansions (3.5) into the Euler equations:

$$g\frac{df_m}{d\lambda}+\left(f-\frac{\varkappa+1}{2}\lambda\right)\frac{dg_m}{d\lambda}+\left(\frac{dg}{d\lambda}+\frac{1}{\lambda}g\right)f_m+$$
$$\left[\frac{df}{d\lambda}+\frac{1}{\lambda}f-\frac{m(\varkappa+1)}{4n}\right]g_m+\frac{k}{\lambda}gu_m=0$$

$$g\left(f-\frac{\varkappa+1}{2}\lambda\right)\frac{df_m}{d\lambda}+\frac{\varkappa-1}{2}\frac{dh_m}{d\lambda}+\left[\frac{df}{d\lambda}+\frac{\varkappa+1}{2n}\left(n-1-\frac{m}{2}\right)\right]gf_m+$$
$$\left[\left(f-\frac{\varkappa+1}{2}\lambda\right)\frac{df}{d\lambda}+\frac{(n-1)(\varkappa+1)}{2n}f\right]g_m=0 \qquad (3.6)$$

$$g\left(f-\frac{\varkappa+1}{2}\lambda\right)\frac{du_m}{d\lambda}-\frac{k(\varkappa-1)}{2\lambda}h_m+\left[\frac{1}{\lambda}f+\frac{\varkappa+1}{2n}\left(n-1-\frac{m}{2}\right)\right]gu_m=0$$

$$\varkappa h\frac{df_m}{d\lambda}+\left(f-\frac{\varkappa+1}{2}\lambda\right)\frac{dh_m}{d\lambda}+\left(\frac{dh}{d\lambda}+\frac{\varkappa}{\lambda}h\right)f_m+$$

$$\left[\varkappa \frac{df}{d\lambda} + \frac{\varkappa}{\lambda} f + \frac{\varkappa+1}{n}\left(n - 1 - \frac{m}{4}\right)\right] h_m + \frac{k\varkappa}{\lambda} h u_m = 0$$

The initial values of the functions f_m, g_m and h_m do not change, they are given by (2.4) with $\nu = 2$. On the basis of the second of the conditions (1.2), we find that the function $u_m = 1$ at the point $\lambda = 1$. The scheme of the subsequent reasoning is completely analogous to that used above in studying the plane waves. The velocity component for a selected direction of reading the angle φ is $v_z = v_r \cos\varphi - v_\varphi \sin\varphi$. By virtue of the symmetry of the initial self-similar flow, the total gas momentum is proportional to the small parameter ε, where only terms with the first harmonic yield a nonzero contribution to the integral (3.1). Hence, let us take $k = 1$ and $\alpha_k = 0$ at once. Then

$$I(\lambda_2, \lambda) = \frac{2n\pi}{\varkappa - 1} \varepsilon \rho_1 b^{3n} t^{3n-1-m/2}\left[1 + \int_\lambda^1 (g f_m + f g_m - g u_m)\lambda\, d\lambda\right] + \ldots$$

This expression does not alter with time if the exponent

$$m = 2(3n - 1)$$

We find the derivative of the momentum by using (3.2). Consequently

$$\frac{dI(\lambda_2, \lambda)}{dt} = -\frac{2n^2\pi}{\varkappa - 1} \varepsilon \rho_1 b^{3n} t^{3n-2-m/2}\left[\lambda^2 (g f_m + f g_m - g u_m) - \frac{2}{\varkappa+1}\lambda\left(2 f g f_m + f^2 g_m - f g u_m + \frac{\varkappa-1}{2} h_m\right)\right] + \ldots$$

Since the total gas momentum is constant in the approximation under consideration, then

$$\lambda(g f_m + f g_m - g u_m) - \frac{1}{\varkappa+1}\left[4 f g f_m + 2 f^2 g_m - 2 f g u_m + (\varkappa - 1) h_m\right] = 0 \quad (3.7)$$

This relationship is the first integral of the system of ordinary differential equations (3.6), it also satisfies the boundary conditions on the surface of strong discontinuity.

In conclusion, let us examine the momentum transfer in waves whose shape differs slightly from the spherical. In this case the parameter $\nu = 3$. The position of the shock wave in three-dimensional flows is given by the two angles φ and ϑ of a spherical coordinate system. We shall read the angle ϑ from the direction of the z-axis. Let us assume that the quantity R_2 from the expansions (1.1) and (2.5) can be represented as a series in the spherical functions Y_l^k, which by definition satisfy the following partial differential equation [7]:

$$\frac{\partial^2 Y_l^k}{\partial \varphi^2} + \sin\vartheta \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial Y_l^k}{\partial\vartheta}\right) + l(l+1)\sin^2\vartheta\, Y_l^k = 0 \quad (3.8)$$

Its solutions consist of products of $\cos k\varphi$ or $\sin k\varphi$ by the associated Legendre function $P_l^k(\cos\vartheta)$. Taking a term with arbitrary number in a series of spherical functions, let us write $R_2 = Y_l^k(\varphi, \vartheta)$. We seek the gas parameters corresponding to this shock front perturbation as

$$v_r = \frac{2n}{\varkappa+1} b^n t^{n-1}\left[f(\lambda) + \varepsilon t^{-{}^1\!/_2 m} f_m(\lambda) Y_l^k(\varphi, \vartheta) + \ldots\right]$$

$$v_\varphi = -\frac{2n}{\varkappa+1} \varepsilon b^n t^{n-1-{}^1\!/_2 m} u_m(\lambda) \frac{1}{\sin\vartheta}\frac{\partial Y_l^k}{\partial\varphi} + \ldots$$

$$v_\vartheta = -\frac{2n}{\varkappa+1}\varepsilon b^n t^{n-1-2/5m} w_m(\lambda)\frac{\partial Y_l^k}{\partial\vartheta} + \ldots \tag{3.9}$$

$$\rho = \frac{\varkappa+1}{\varkappa-1}\rho_1 [g(\lambda) + \varepsilon t^{-2/5m} g_m(\lambda) Y_l^k(\varphi, \vartheta) + \ldots]$$

$$p = \frac{2n^2}{\varkappa+1}\rho_1 b^{2n} t^{2(n-1)} [h(\lambda) + \varepsilon t^{-2/5m} h_m(\lambda) Y_l^k(\varphi, \vartheta) + \ldots]$$

Now, let us add terms containing derivatives with respect to φ, ϑ and the velocity vector components v_φ, v_ϑ to the Euler equations (2.2) in order to obtain the complete system of equations of gasdynamics in spherical cooddinates. Such a transformation does not influence the determination of the functions f, g and h giving symmetric self-similar flow. Upon substituting the expansions (3.9) into the Euler equations, it is seen first that the functions u_m and w_m are solutions of the same ordinary differential equation. The initial values of the functions f_m, g_m and h_m are prescribed, as before, by (2.4) with $\nu = 3$. As regards the initial values of the functions u_m and w_m, they are obtained from the condition of conservation of the tangential velocity component upon passage through the wave shock front. This condition decomposes into two independent relations from which it follows that $u_m = 1$ and $w_m = 1$ at the point $\lambda = 1$. As is now clear, $u_m = w_m$ in the whole range of variation of λ. The remaining four equations for the functions f_m, g_m, h_m and w_m are

$$g\frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dg_m}{d\lambda} + \left(\frac{dg}{d\lambda} + \frac{2}{\lambda}g\right)f_m + \left[\frac{df}{d\lambda} + \frac{2}{\lambda}f - \frac{m(\varkappa+1)}{5n}\right]g_m + \frac{l}{\lambda}gw_m = 0$$

$$g\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{df_m}{d\lambda} + \frac{\varkappa-1}{2}\frac{dh_m}{d\lambda} + \left[\frac{df}{d\lambda} + \frac{\varkappa+1}{2n}\left(n-1-\frac{2m}{5}\right)\right]gf_m + \left[\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{df}{d\lambda} + \frac{(n-1)(\varkappa+1)}{2n}f\right]g_m = 0 \tag{3.10}$$

$$g\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dw_m}{d\lambda} - \frac{\varkappa-1}{2\lambda}h_m + \left[\frac{1}{\lambda}f + \frac{\varkappa+1}{2n}\left(n-1-\frac{2m}{5}\right)\right]gw_m = 0$$

$$\varkappa h\frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dh_m}{d\lambda} + \left(\frac{dh}{d\lambda} + \frac{2\varkappa}{\lambda}h\right)f_m + \left[\varkappa\frac{df}{d\lambda} + \frac{2\varkappa}{\lambda}f - \frac{\varkappa+1}{n}\left(n-1-\frac{m}{5}\right)\right]h_m + \frac{l\varkappa}{\lambda}hw_m = 0$$

Let us proceed to a computation of the total gas momentum. Since the initial self-similar flow possesses spherical symmetry, it should be on the order of ε, where the non-zero contribution to the integral (3.1) is stipulated by terms independent of the longitude φ. Hence $k = 0$, and (3.8) is transformed into an ordinary differential equation for the Legendre polynomials $P_l(\cos\vartheta)$. If the orthogonality of the Legendre polynomials and the equality $P_1(\cos\vartheta) = \cos\vartheta$ are taken into account, then it is clear that we can set $l = 1$ and $Y_1^0 = \cos\vartheta$ in calculating the total gas momentum. Substituting the relationship $v_z = v_r\cos\vartheta - v_\vartheta\sin\vartheta$ into the integral (3.1), we have

$$I(\lambda_2, \lambda) = \frac{8n\pi}{3(\varkappa-1)}\varepsilon\rho_1 b^{4n} t^{4n-1-2m/5}\left[1 + \int_\lambda^1 (gf_m + fg_m - 2gw_m)\lambda^2 d\lambda\right] + \ldots$$

This expression remains constant for

$$m = {}^5/_2 (4n - 1)$$

As before, we find the derivative of the momentum by using (3.2), from which

$$\frac{dI(\lambda_2, \lambda)}{dt} = -\frac{8n^2\pi}{3(\varkappa - 1)} \varepsilon \rho_1 b^{4n} t^{4n-2-2m/5} \Big[\lambda^3 (g f_m + f g_m - 2g w_m) - \frac{2}{\varkappa + 1} \lambda^2 \Big(2fg f_m + f^2 g_m - 2fg w_m + \frac{\varkappa - 1}{2} h_m\Big)\Big] + \dots$$

For the selected value of m the derivative $dI(\lambda_2,\lambda) / dt$ should vanish. The mentioned condition results in the integral

$$\lambda (g f_m + f g_m - 2g w_m) - \frac{1}{\varkappa + 1} [4fg f_m + 2f^2 g_m - 4fg w_m + (\varkappa - 1) h_m] = 0 \quad (3.11)$$

of the system (3.10), which automatically satisfies the relationships on the shock wave front. Let us note that (3.4), (3.7) and (3.11) can be given the standard form of the first integrals of the ordinary differential equations considered above by replacing the zeros in their right sides by the quantities c_I, $c_I\lambda^{-1}$ and $c_I\lambda^{-2}$, respectively, where the constant c_I is arbitrary.

4. The integrals which are connected with the mass, energy and momentum conservation laws in the perturbed motion zone, are applicable to the solution of many problems of gasdynamics. Let us present two simple illustrations.

First, let a piston expand in time according to a power-law in a gas where liberation of a finite quantity of energy occurred beforehand. It is clear that it can be taken into account by using the expansion (2.1) with index n corresponding to the piston motion law, and the parameter $m = {}^1/_2 (\nu + 2) [(\nu + 2) n - 2]$.

As a second illustration, let us consider the effect of momentum on the nonstationary flow originating in a strong explosion. As shown in [5, 6], $n = 2/(\nu + 2)$ in the main self-similar solution. As regards the exponent m governing the variational function, in this case

$$m = {}^1/_2 (\nu + 2) [(\nu + 1) n - 1] = {}^1/_2 \nu$$

One more remark should still be made about higher approximations in the method of perturbations. If the system of ordinary differential equations for the additional terms with some number is homogeneous, then it possesses the integrals established above, which retain their form completely. The reasoning is easily extended to inhomogeneous systems also, where an independent calculation of the mass, energy, and momentum of the gas and their time-derivatives permits taking account of the presence of right sides in the differential equations.

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GROUP CLASSIFICATION OF EQUATIONS OF HYDRODYNAMICS OF A PERFECT FLUID

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Equations of perfect fluid hydrodynamics are classified with respect to the Coriolis parameter, and all essentially different solutions of rank one are indicated.

1. Statement of the problem. Let x and y be Cartesian coordinates, u and v the velocity components along the x and y axes, respectively, and p the pressure; the density ρ is assumed constant and equal unity. We consider systems of equations of the form

$$uu_x + vu_y - lv = -p_x, \quad uv_x + vv_y + lu = -p_y, \quad u_x + v_y = 0 \qquad (1.1)$$

in which the parameter $l(y)$ can be an arbitrary function of y. For an arbitrary $l(y)$ system (1.1) admits a certain group of transformations G. The special forms of function $l(y)$ for which the fundamental group admitted by system (1.1) is wider than G are to be determined.

Equations (1.1) are encountered in meteorological problems in which the terms lu and lv represent components of acceleration produced by the Coriolis force owing to the rotation of Earth, and $l(y)$ is the Coriolis parameter. For $l = 0$ system (1.1) coincides with that of the usual equations of hydrodynamics of a perfect fluid. The determination of the group for this case is given in [1] on the assumption of unsteady flow.

Besides the determination of the group of transformations we shall derive solutions of rank one, i.e. such which reduce their derivation to the integration of ordinary differential equations. Some of these solutions were obtained earlier, for instance, in [2] solutions with spiral streamlines are indicated. In the present paper the problem of group classification of system (1.1) is solved, optimal systems of one-parameter subgroups are determined, and all essentially different solutions are indicated. Since the required mechanism of group analysis is presented in [3], many intermediate computations are omitted.

2. Classification of equations. To calculate the coordinates of the infinitesimal operator of the group admitted by system (1.1) it is necessary to write out the so-called defining equations and to solve these.

1) For any arbitrary function $l(y)$ the basic operators of the related Lie algebra are of the form

$$X_1 = \partial / \partial p, \qquad X_2 = \partial / \partial x \qquad (2.1)$$

The analysis of determining equations for other forms of function $l(y)$ yields the